18.100A PSET 3 SOLUTIONS

DAVID CORWIN

Problem 1

(a). If $\sum a_n$ is absolutely convergent, then it is convergent, so $a_n \to 0$ as $n \to \infty$. Thus, there is N such that for $n > N$, we have $|a_n| < 1$. It follows that for such n, we have $|a_n^2| = |a_n|^2 \leq |a_n|$.

By tail convergence, we know that $\sum_{n>N} |a_n|$ converges, so by the comparison theorem for positive series, we find that $\sum_{n>N} |a_n^2| = \sum_{n>N} a_n^2$ converges. Again, by tail convergence, this implies that $\sum_n a_n^2$ converges.

(b). We consider $a_n = \frac{(-1)^n}{\sqrt{n}}$. By Cauchy's test for alternating series, this converges. However, a_n^2 is the harmonic series, which is known to diverge.

Problem 2

For each n, set

$$a_n^+ = \frac{|a_n| + a_n}{2}$$
$$a_n^- = \frac{|a_n| - a_n}{2}$$

Then $a_n = a_n^+ - a_n^-$ for all n, and $a_n^+, a_n^- \geq 0$.

Suppose that a_n has finitely many positive terms. Then $a_n^+ = 0$ for all but finitely many n, so the series $\{a_n^+\}$ converges. It follows that $a_n^- = a_n^+ - a_n$

converges, hence so does $|a_n| = a_n^+ + a_n^-$, a contradiction to conditional convergence.

Suppose that a_n has finitely many negative terms. Then $a_n^- = 0$ for all but finitely many n, so the series $\{a_n^-\}$ converges. It follows that $a_n^+ = a_n - a_n^-$ converges, hence so does $|a_n| = a_n^+ + a_n^-$, a contradiction to conditional convergence.

In either case, we see that if a_n converges conditionally (i.e., $|a_n|$ does not converge), then a_n either has infinitely many positive or infinitely many negative terms.

Problem 3

(b). Setting $a_n = \frac{n^2}{2^n}$, we have

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)^2/2^{n+1}}{n^2/2^n}\right| \\ &= \frac{1}{2}\left(\frac{n+1}{n}\right)^2\end{aligned}$$

Thus

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1.$$

So by the ratio test, this series converges.

(d). Setting $a_n = \frac{(n!)^2}{(2n)!}$, we have

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)!^2/(2n+2)!}{(n!)^2/(2n)!}\right| \\ &= \frac{(n+1)!^2/(n!)^2}{(2n+2)!/(2n)!} \\ &= \frac{(n+1)^2}{(2n+2)(2n+1)} \\ &= \frac{n^2+2n+1}{4n^2+6n+2} \\ &= \frac{1+2/n+1/n^2}{4+6/n+2/n^2}\end{aligned}$$

Thus
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1+2/n+1/n^2}{4+6/n+2/n^2} = \frac{1}{4} < 1.$$

So by the ratio test, this series converges.

(j). By the integral test, we may compare this with $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$.

For $= p \neq 1$, the corresponding indefinite integral is $\dfrac{(\ln x)^{1-p}}{1-p}$. We thus have
$$\int_2^\infty \frac{dx}{x(\ln x)^p} = \left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^\infty$$

As $\lim_{x\to\infty} \ln x = \infty$, this converges only when $p > 1$.

Finally, if $p = 1$, the corresponding indefinite integral is $\ln\ln x$. We thus have
$$\int_2^\infty \frac{dx}{x\ln x} = [\ln\ln x]_2^\infty$$

As $\lim_{x\to\infty} \ln x = \infty$, we also have $\lim_{x\to\infty} \ln\ln x = \infty$, so the integral diverges.

In summary, we have convergence only when $p > 1$.

Problem 4

Let's suppose that the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists. Then if we apply the ratio test to $\sum a_n x^n$, we are considering the limit
$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

This is less than one iff
$$|x| < \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)^{-1},$$
so
$$R = \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)^{-1} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

(a). Setting $a_n = \dfrac{1}{2^n\sqrt{n}}$, we have

$$\begin{aligned}\left|\frac{a_n}{a_{n+1}}\right| &= \left|\frac{1/(2^n\sqrt{n})}{1/(2^{n+1}\sqrt{n+1})}\right| \\ &= \left|\frac{2^{n+1}\sqrt{n+1}}{2^n\sqrt{n}}\right| \\ &= 2\sqrt{1+\frac{1}{n}}\end{aligned}$$

Thus

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} 2\sqrt{1+\frac{1}{n}} = 2.$$

Therefore, the ratio of convergence is 2.

(f). The nth root of the nth term is $\dfrac{x}{\ln n}$. For all x, we have $\lim_{n\to\infty}\dfrac{x}{\ln n} = 0$, so by the nth root test, we see that the nth root of the nth term approaches 0. It follows that the series converges for all x, i.e., the radius of convergence is ∞.

Problem 5

Let $f(x) = \dfrac{x}{1+x}$. We have $f(1) = \dfrac{1}{2}$, so we need to show that $\lim_{x\to 1} f(x) = \dfrac{1}{2}$.

Given $\epsilon > 0$, let $\delta = \min(1, \epsilon)$.

Then if $|x-1| < \delta$, we have

$$\begin{aligned}\left|f(x) - \frac{1}{2}\right| &= \left|\frac{x}{1+x} - \frac{1}{2}\right| \\ &= \left|\frac{2x}{2+2x} - \frac{1+x}{2+2x}\right| \\ &= \left|\frac{x-1}{2+2x}\right| \\ &= \frac{|x-1|}{|2+2x|}\end{aligned}$$

As $|x-1| < \delta \le 1$, we have $x > 0$, so $|2+2x| > 2$, so

$$\left|f(x) - \frac{1}{2}\right| = \frac{|x-1|}{|2+2x|} \le |x-1| < \delta \le \epsilon.$$

As $\epsilon > 0$ was arbitrary, we are done.

Problem 6

We have $\sin^2 x = 1 - \cos^2 x = (1-\cos x)(1+\cos x)$, so

$$1 - \cos x = \frac{\sin^2 x}{1+\cos x},$$

unless $\cos x = -1$. But $\cos 0 = 1$, so $\cos x \ne -1$ for x in a neighborhood of 0.

We therefore have

$$\begin{aligned} \lim_{x\to 0} \frac{1-\cos x}{x} &= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} \\ &= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{\sin x}{1+\cos x}\right) \\ &= (1)\left(\frac{0}{1+1}\right) \\ &= 0. \end{aligned}$$

Problem 7

As the function contains $\sqrt{x}$, we are only considering $x \ge 0$. This will be assumed implicitly in all that follows.

For all $x \ne 0$, we have $|\cos(1/x)| \le 1$. It follows that for $x \ne 0$, we have $|f(x)| = \sqrt{x}|\cos 1/x| \le \sqrt{x}$. Thus $0 \le |f(x)| \le \sqrt{x}$, so by the squeeze theorem for limits, $0 \le \lim_{x\to 0} |f(x)| \le \lim_{x\to 0} \sqrt{x} = 0$. It follows that $\lim_{x\to 0} |f(x)| = 0$, hence also $\lim_{x\to 0} f(x) = 0 = f(0)$, so f is continuous at 0.

Problem 8

(a). Suppose there were x_0 such that $f(x_0) \ne 0$. Let $\epsilon = |f(x_0)|/2 > 0$. Then there is $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for $x \in (x_0 - \delta, x_0 + \delta)$.

In particular, for such x, we have $f(x) > |f(x_0)|/2 > 0$. But we can find a rational number r in $(x_0 - \delta, x_0 + \delta)$, so $f(r) = 0$, a contradiction.

(b). Again, suppose there were x_0 such that $f(x_0) > g(x_0)$. Let $\epsilon = |f(x_0) - g(x_0)|/2 > 0$. Then there is $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ and $|g(x) - g(x_0)| < \epsilon$ for $x \in (x_0 - \delta, x_0 + \delta)$. In particular, for such x, we have $f(x) > f(x_0) - \epsilon = \dfrac{f(x_0) + g(x_0)}{2}$, and $g(x) < g(x) + \epsilon = \dfrac{f(x_0) + g(x_0)}{2}$. But we can find a rational number r in $(x_0 - \delta, x_0 + \delta)$, so $f(r) \leq g(r)$, a contradiction.

As a counterexample, take $f(x) = 0$ and $g(x) = (x - \sqrt{2})^2$. Then $f(x) < g(x)$ for all rational x, but $f(\sqrt{2}) = g(\sqrt{2}) = 0$.

Problem 9

Let us take $x_n = \dfrac{n\pi}{2}$. Assume that $\lim_{x\to\infty} \sin x$ exists.

Applying Theorem 11.5A for $a = \infty$ (if one were worried about the theorem applying with $a = \infty$, one could also apply it to $\sin(1/x)$ with $a = 0$), we find that since $\lim_{n\to\infty} x_n = \infty$, the limit $\lim_{n\to\infty} \sin x_n$ also exists. Call this limit L.

Taking the subsequence x_{2n}, we have

$$\lim_{n\to\infty} \sin x_{2n} = \lim_{n\to\infty} \sin(2\pi n) = \lim_{n\to\infty} 0 = 0,$$

so $L = 0$. Taking the subsequence x_{4n+1}, we have

$$\lim_{n\to\infty} \sin x_{4n+1} = \lim_{n\to\infty} \sin\left(2\pi n + \frac{\pi}{2}\right) = \lim_{n\to\infty} 1 = 1,$$

so $L = 1$.

This is a contradiction, so $\lim_{x\to\infty} \sin x$ does not exist.

Problem 10

If f is multiplicatively periodic with constant c, we note that $f(x) = f(cc^{-1}x) = f(c^{-1}x)$, so f is multiplicatively periodic with constant c^{-1}. If $c > 1$, then $c^{-1} < 1$, so we may assume that f is multiplicatively periodic for a constant less than one. In other words, without lack of generality, we may assume $c < 1$.

Applying the relation $f(x) = f(cx)$ iteratively, we find that $f(x) = f(c^n x)$ for any x.

Consider the sequence $\{c^n x\}$. This sequence has limit 0 as $n \to \infty$. Thus $\lim_{n\to\infty} f(c^n x) = f(0)$ by continuity of f. But $f(c^n x) = f(x)$, so this limit is also $\lim_{n\to\infty} f(c^n x) = \lim_{n\to\infty} f(x) = f(x)$. Thus $f(x) = f(0)$ for all x, so the function is constant.